SIGNALS & SYSTEMS

DOCUMENTATION OF VIDEO LECTURES

K. S. Venkatesh
IIT-Kanpur

SIGNALS

- Signals are changes of any quantity w.r.t. time, space or both.
- Signals can be represented by functions.
- Functions are defined as maps or associations between objects (Domain & Range).
- Thus signals can be represented as maps from domain to range.
- Commonly used Domain set:
  - Set of Integers
  - Set of real numbers
- Commonly used Range set:
  - Set of complex numbers

DOMAIN & RANGE OF SIGNALS

- Both Domain and Range can be multidimensional.
- Signals where the domains are of higher dimension (greater than 1), are called multidimensional signals.
- Signals where the range are of higher dimension (greater than 1), are called multivariate signals.
- Range set contains all possible values that the function can assume.
- Complex numbers comprises Range set.

Theory of Complex numbers

- Rectangular representation:

$$c = a + jb \quad : \quad j = \sqrt{-1}$$

$$a = Real\,[c] \& \; b = Imaginary\,[c]$$

- Polar representation:

$$c = r\angle\theta \qquad : r = \sqrt{a^2 + b^2}$$

$$a = r\sin\theta \;;\; b = r\cos\theta \;\&\; \theta = \tan^{-1}(\tfrac{a}{b})$$

Operations on Complex numbers:

- Addition:

$$c + c' = (a + a') + j(b + b')$$

- Subtraction:

$$c + (-c') = (a - a') + j(b - b')$$

- Multiplication:

$$c.c' = (ab - a'b') + j(a'b + ab') : rectangular\ representation$$

$$c.c' = r.r' \angle \theta + \theta' \qquad : polar\ representation$$

- Division:

$$\frac{c}{c'} = \frac{r}{r'} \angle \theta - \theta'$$

- Raising to a power:

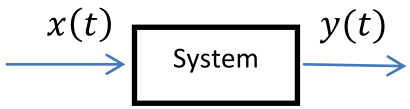
$$c^n = r^n \angle n\theta$$

- Extracting n^{th} root of complex number:
  - $\sqrt[n]{c}\ will\ have\ n\ different\ n^{th} roots\ of\ c$

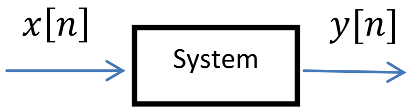
$$\sqrt[N]{c} = \sqrt[N]{r}\angle\varphi_1, \sqrt[N]{r}\angle\varphi_2, \dots \sqrt[N]{r}\angle\varphi_n, \dots\dots\dots \sqrt[N]{r}\angle\varphi_N \ : \ \varphi_n = \frac{\theta + (n-1)2\pi}{n};\ \mathrm{n} = 1, \dots, \mathrm{N}$$

SYSTEM INTRODUCTION

- System is an entity that takes a signal and converts into another signal.
- A system is continuous time if its input-output signal is continuous.



- A system is discrete time if its input-output signal is discrete.



- Continuous time system processes continuous time signals $x(t)$ and produces continuous-time outputs $y(t)$.
- Discrete time system processes discrete time signals $x[n]$ and produces discrete-time outputs $y[n]$.
- Different systems differ in the manner in which they relate the inputs and outputs.
- A system is also a function as it maps input to output.
- A map is Injective (*one-one*) if it takes different input signals to different output signals.

$$If\ x_1 \neq x_2\ then\ y_1 \neq y_2$$

- A map is Surjective (*onto*) if every signal in the range set will be found as the output for some input signal.
- A system can be either Injective or Surjective, or both or neither.

SIGNAL PROPERTIES

- A signal must be defined for every value of the argument.
- Signals that have finite value at every instant of time are finite valued signals.
- Signals that have values within a certain limit are called bounded signals while signals for which bounds cannot be found are called unbounded signals.
- All physical signals are inherently bounded because any physical source of signals can produce only limited signal power.
- Symmetries are intimately related to signal representations.
- Signals can have following symmetry properties:
  - Periodicity: $x(t) = x(t - T)$
  - Even symmetry: $x(t) = x(-t)$
  - Odd symmetry: $x(t) = -x(-t)$
- A signal can be decomposed into even and odd parts : $x(t) = x_e(t) + x_o(t)$

FREQUENTLY USED CONTINUOUS TIME SIGNALS

- $u_k(t)$ Family :
    - Unit step $u_{-1}(t) = u(t) = \begin{cases} 1 : t \geq 0 \\ 0 : t < 0 \end{cases}$
    - Unit ramp $u_{-2}(t) = r(t) = \begin{cases} t : t \geq 0 \\ 0 : t < 0 \end{cases}$
- Exponential Family:e^{st} $(s = \sigma + j\omega)$
    - Real exponential signals $x(t) = e^{\sigma t}$
    - Complex (imaginary) exponential signals $x(t) = e^{j\omega t}$
    - General complex exponential signals $x(t) = e^{(\sigma + j\omega)t}$

FREQUENTLY USED DISCRETE TIME SIGNALS

- $u_k[n]$ Family :
  - Unit step $u_{-1}[n] = u[n] = \begin{cases} 1 : n \geq 0 \\ 0 : n < 0 \end{cases}$
  - Unit ramp $u_{-2}[n] = r[n] = \begin{cases} n : n \geq 0 \\ 0 : n < 0 \end{cases}$
- Discrete Exponential Family: a^n $(a = re^{j\Omega})$
  - Real exponential signals $x[n] = r^n$ $: \Omega = 0$
  - Complex (imaginary) exponential signals $x[n] = e^{j\Omega n}$ $: r = 1 \;\&\; \Omega > 0$
  - General complex exponential signals $x[n] = r^n . e^{j\Omega n}$

TRANSFORMATION ON DOMAIN & RANGE

- A system is viewed as transformation or mapping of input to output.

Transformation on Time:

- On input signal $x(t)$ (defined on time), application of function on time and then a rearrangement of time axis.
- Examples of transformation:
    - Translation: $f(t) = t + \tau$
    - Flip: $f(t) = -f(t)$
    - Scaling: $f(t) = at$, where a is a constant for $a \neq 0$

Transformation on Range:

- Transformation of a signal in the range.
- Types of transformation:
    - Translation (adding a constant): $f(t) = f(t) + a$, where a is a constant
    - Flip: $f(t) = -f(t)$
    - Scaling: $f(t) = af(t), a \neq 0$
- Range and Time (Domain) can be combined.
- Interchange of range and domain components of a complete transformation does not affect the end result.
- Change in internal order of the domain or range transformation, changes the end result.

SYSTEM PROPERTIES

- A system is a mathematical model of a physical process that relates input (excitation) to output (response).

Properties of system:

- Memory or Memory less system
  - System is memoryless if its output $y(t_0)$ depends only on its input $x(t_0)$ at that same instant t_0.
  - Attenuators and amplifiers are examples of memoryless systems.
- Causality
  - System is causal if its output $y(t)$ at t_0 only depends upon input $x(t) : t \leq t\_0$
  - System is anti-causal if its output $y(t)$ at t_0 depends only upon input $x(t) : t > t_0$
  - System is non-causal or acausal if $y(t)$ at t_0 depends only upon $x(t) : -\infty < t - t_0 < \infty$
- All memoryless systems are causal but not vice versa.

SYSTEM PROPERTIES

- Stability
  - A system is BIBO (Bounded Input/Bounded Output) stable if any bounded input $x(t)$ results in a bounded output $y(t)$

$$|x(t)| \leq B_x \ : \ -\infty < t < \infty \ \rightarrow |y(t)| \leq B_y \ : \ -\infty < t < \infty$$

$B_x \ and \ B_y$ are finite and real.

- Linearity
  - A system is linear if it satisfies Homogeneity and Additivity.
  - Homogeneity:

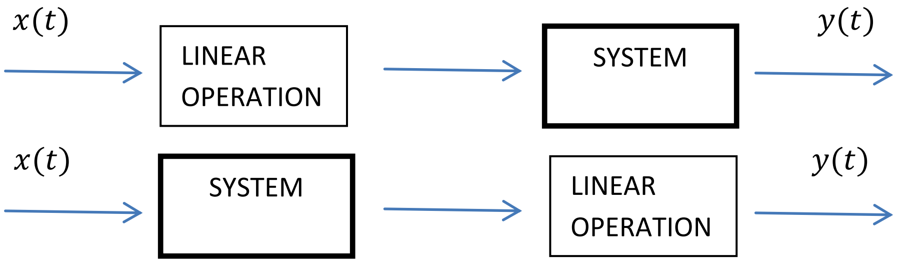
$$x(t) \rightarrow y(t) \Rightarrow kx(t) \rightarrow ky(t) \ : k \ is \ complex$$

  - Additivity:

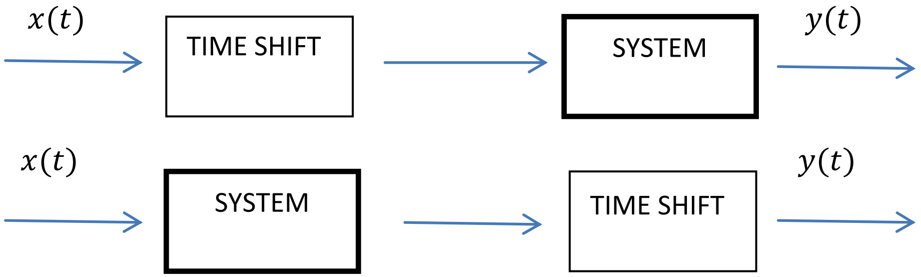
$$x(t) \rightarrow y(t) \ \& \ x^{'}(t) \rightarrow y^{'}(t)$$
$$x(t) + x^{'}(t) \rightarrow y(t) + y^{'}(t)$$

COMMUTATION DIAGRAM AS A TEST FOR LINEARITY & TIME INVARIANCE

- Property of linearity if possessed by a system, is equivalent to a commutation of the linear operations with the system itself.
- Commutation diagram as a test for Linearity:



- Commutation diagram as a test for Time Invariance:

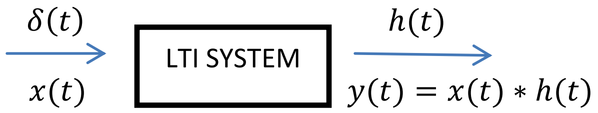


- Every system for which outputs are invariant to the order in which linear (or time shift) operation and system are applied are said to be linear (or Time Invariant).

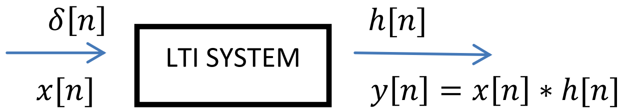
LTI SYSTEMS

- An LTI system is both linear and time invariant.
- Representation of LTI system

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad : \; Continuous - time\ system$$



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad : Discrete - time\ system$$



REPRESENTATION OF DISCRETE TIME CONVOLUTION

- Impulse representation of an input x[n]:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Convolution:

- For Discrete-time signals x[n] and h[n], their convolution is given as:

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- The ouput of a discrete LTI system is the convolution of input x[n] with the impulse response h[n].

Properties of Convolution:

- Commutative:

$$h[n] * x[n] = x[n] * h[n]$$

- Distributes over addition:

$$[x_1[n] + x_2[n]] * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

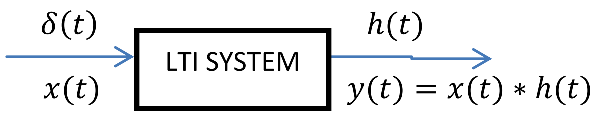
- Associative:

$$[x_1[n] * x_2[n]] * h[n] = x_1[n] * [x_2[n] * h[n]]$$

REPRESENTATION OF CONTINUOUS TIME CONVOLUTION

- Impulse representation of a continuous LTI system for input x[n]:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$



- For Continuous-time signals x(t) and h(t), their convolution is given as:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- The ouput of a discrete LTI system is convolution of input x[n] with the impulse response h[n].

PROPERTIES OF CONTINUOUS-TIME CONVOLUTION

Properties of Convolution:

- Commutative:

$$x(t) * h(t) = h(t) * x(t)$$

- Distributes over addition:

$$[x_1(t) + x_2(t)] * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$$

- Associative:

$$[x_1(t) * x_2(t)] * h(t) = x_1(t) * [x_2(t) * h(t)]$$

Evaluation of system properties from the impulse response:

- Memory less:

$$h(t) = k\delta(t)$$

- Causal:

$$h(t) = 0 \ \ : \ t < 0$$

- Stable:

$$\int_{-\infty}^{\infty} | h(\tau)| d\tau \ < \infty$$

DIFFERENTIAL EQUATION

INTRODUCTION:

- Continuous-time, linear, time invariant, dynamic systems are described by linear ordinary differential equation with constant coefficients.
- Mathematical models of such systems:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = x(t)$$

$x(t)$: forcing function or system input

$y(t)$: solution or system output

- The solution of a linear differential equation $y(t)$ has two additive components:
  - ➢ Homogeneous Solution $y_h(t)$
  - ➢ Non Homogeneous Solution $y_{nh}(t)$

$$y(t) \rightarrow y_h(t) + y_{nh}(t)$$

SOLVING DIFFERENTIAL EQUATION

- The homogeneous solution is contributed by the auxiliary conditions of the system.
- A homogeneous solution satisfies the homogeneous differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

- Solving homogeneous equation: The space of homogeneous solutions has a dimension equal to the order of the differential equation.
- The Non-homogeneous solution is contributed by the forcing function of the system.
- A Non-homogeneous solution satisfies the Non-homogeneous differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = x(t)$$

- Solving Non-homogeneous Equation: This follows from the recognition of classes of functions that remain invariant to differentiation and additive combination. Functions such as constants, polynomials and exponentials. Other arbitrary forcing functions need to be expressed in terms of these invariant functions.

PHYSICAL SYSTEM AND DIFFERENTIAL EQUATIONS

- In addition to the external forcing function, the system is also driven by its internal forces.
- This internal force comes from the system's initial / auxiliary conditions.
- Multiple solution trajectories result from the mathematical solution of differential equations of the system.
- In order to identify from the multiple solutions, the unique solution that corresponds to the actual physical system:
  - The description provided by the differential equation is used.
  - The additional information outside the scope of differential equation is required to be supplied.
  - The additional information can be supplied in a variety of forms at instants of time of our choosing.
  - If the additional information is adequate, then a unique solution will be deduced, that matches with the behavior of the system.

SYSTEM DESCRIBED BY DIFFERENTIAL EQUATION

Linearity:

- A system is said to be linear if it meets the following two criteria:
  - Homogeneity: $kx(t) \rightarrow ky(t)$
  - Additive: $x_1(t) \rightarrow y_1(t) \;\&\; x_2(t) \rightarrow y_2(t)$
    $x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$

The linear behavior of system described by a differential equation:

- The response of a system is linear against input if all the auxiliary conditions are zero at the reference instant.
- The response of a system is linear w.r.t. any one auxiliary condition when input and all the other auxiliary conditions are set to zero.
- Since response consists of two components:
  - One component is linear w.r.t. Input : $y_I(t)$
  - Other component is similarly linear w.r.t. to auxiliary condition: $y_A(t)$

$$y(t) \rightarrow y_A(t) + y_I(t)$$

: $y(t)$= Output/response

: $y_A(t)$ =Response when input is zero or Zero input response

: $y_I(t)$= Response when auxiliary condition is zero or Zero state response

- The complete response is neither linear w.r.t. to auxiliary condition or Input if the other is also present..

SYSTEM DESCRIBED BY DIFFERENTIAL EQUATION

Time Invariance:

$$x(t) \rightarrow y(t)$$

$$x(t-t_0) \rightarrow y(t-t_0) \begin{cases} for\ all\ t_0 & ;\ \infty < t_0 < \infty \\ for\ all\ x(t) \end{cases}$$

The Time Invariance behavior of system described by a differential equation:

- A system described by a differential equation is time invariant when the time of specification of the auxiliary conditions is shifted by the same amount by which we shift the input signal.
- Alternatively, it is time invariant if, instead of shifting the auxiliary condition(s), we modify their values to those that they would assume at the new shifted position when we shift the input.

Causality:

- The differential equation itself does not compel the system it describes to be either causal or anti causal.
- The Differential Equation is neutral with respect to the direction of time and can be solved in either direction as we choose.

DIFFERENCE EQUATION

- It is the counterpart of the differential equation for describing discrete-time systems.
- The general form of the linear constant coefficient difference equation:

$$\sum_{k=0}^{n} a_k y[n-k] = x[n]$$

To solve a Linear constant coefficient ordinary difference equation:

The solution of a linear difference equation y[n] has two additive components:

- Homogeneous Solution$y_h[n]$
- Non Homogeneous Solution $y_{nh}[n]$

$$y[n] \to y_h[n] + y_{nh}[n]$$

- A homogeneous difference equation: $\sum_{k=0}^{n} a_k y[n-k] = 0$
- Solving the homogeneous equation:
  - Substituting $y[n]$ for ar^n.
  - Factor out the exponential ar^n, so one is left with the algebraic characteristic equation.
  - Solve the algebraic characteristic equation and find the roots.
  - For N distinct roots, the solution takes the form

$$y[n] = \sum_{k=1}^{N} c_k r_k^n$$

- A non-homogeneous difference equation: $\sum_{k=0}^{n} a_k y[n-k] = x[n]$
- Solving Non-homogeneous Equation:
  - Forcing functions for which we attempt to solve for non-homogeneous equation should be constant/exponential /polynomial.
  - The solution of non-homogeneous equation for the above kinds of forcing function will proceed by assuming that the solution $y[n]$ is a sequence of the same type.
  - The basic solution of the non-homogeneous equation is denoted by $\psi_0[n]$
- Thus the complete solution takes the form:
  $y[n] = \psi_0[n] + \sum_{k=1}^{N} c_k r_k^n \qquad ; \ c_k\text{:arbitrary constant}$
- The relationships pertaining to the specification and insertion of the auxiliary conditions to obtain a unique physical solution follow closely those discussed already for the case of the differential equations.

LTI SYSTEMS DESCRIBED BY DIFFERENCE EQUATION

Analogous to the continuous time case, the properties of a system can be developed based on approach of differential equation.

Linearity:

- The Response of a system is linear against input if the auxiliary condition is zero at the reference instant.
- The Response of a system is linear w.r.t. auxiliary condition when input is set to zero.

$$y[n] \rightarrow y_A[n] + y_I[n]$$

: $y(t)$ $=$ Output/response
: $y_A(t)$ =Response when input is zero or zero input response
: $y_I(t)$ =Response when auxiliary condition is zero or zero state response.

- Combination is neither linear w.r.t. to auxiliary condition or Input

Incremental Linearity

- In the presence of nonzero auxiliary conditions, the output is not linear w.r.t the input as mentioned earlier for differential equations. But the system exhibits 'incremental' linearity: increments of the output are linear with respect to increments of the input. This is because of the cancellation of the output component due to the auxiliary conditions.

$$x_1(t) \to y_1(t) \ : \ y_1(t) = y_A(t) + y_{I1}(t)$$
$$x_2(t) \to y_2(t) \ : \ y_2(t) = y_A(t) + y_{I2}(t)$$
$$\propto [x_2(t) - x_1(t)] \ \to \ \propto [y_{I2}(t) - y_{I1}(t)]$$

Time Invariance

$$x[n - n_0] \to y[n - n_0]$$

- A system described by a difference equation is time invariant when the time of specifications of the auxiliary conditions are shifted by the same amount by which we shift the input signal.
- Alternatively it is time invariant if shifting of the auxiliary condition values that we expect to find at those same time(after shifting is done) instants, is done.

Causality

- The difference equation is neutral with respect to the direction of time, can be solved both causally or anticausally.
- Real physical systems are always causal.

FILTERS

Moving Average Filter

- It essentially creates a gradually moving average, at every instant of time the window over which the input sequence is being averaged is being shifted by one position, along with the point of interest.
- It is called a Finite Impulse Response (FIR) filter as (the support of) its impulse response is finite.
- The order of filter depends on the number of delay blocks.
- The general form of an M^{th} order moving filter : $\sum_{k=0}^{M} b_k\, x[n-k]$
- The finite impulse response of Moving Average filter : $h[n] = \sum_{k=0}^{M} b_k\, \delta[n-k]$

Auto Regressive Filters:

- For Auto Regression filters of order N, the system output at time n is a linear combination of N past outputs.
- It is called Infinite Impulse Response (IIR) filter as its impulse response is invariably (of) infinite (support).
- The general form of an Auto Regressive filter : $\sum_{k=0}^{\infty} a_k\, y[n-k]$

Auto Regression Moving Average (ARMA) filters:

- It is combination of Auto Regression (IIR) & Moving Average (FIR) filters.
- The general form of IIR & FIR filter combination : $\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$

Filter Implementation:

- ➢ Direct Form I
- ➢ Direct form II

IMPLEMENTATION WITH INTEGRATORS

- Integrators are an alternative to differentiators for continuous time implementation of the solver of differential equation.
- Integrators average out minor fluctuations or noise unlike differentiator which enhances them with repeated differentiation.

Solving the differential Equation using Integrator:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}$$

- Equation after Integration over N time : $\sum_{k=0}^{N} a_k y^{[n-k]}(t) = \sum_{k=0}^{N} b_k x^{[n-k]}(t)$
- Final equation : $y(t) = \sum_{k=0}^{N} b_k\, x^{[n-k]}(t) - \sum_{k=0}^{N-1} a_k y^{[n-k]}(t)$

Filter Implementation:

- Direct Form I
- Direct form II

- For a causal implementation of a difference/differential equation solver, the auxiliary condition necessarily has to be the initial condition. For an anti-causal system being solved for $t \rightarrow -\infty\ to\ t = 0$, the auxiliary condition has necessarily to be a final condition.

THEORY OF SIGNAL REPRESENTATION

- The motivation for signal representation is the convenience of dealing with a small number of signals taken out of a fixed collection.
- This provides a much more compact description of the systems behavior.
- Instead of dealing with a processor in terms of what it does to each possible signal in the signal space, it is much more convenient to decompose the signal space into small classes.
- Each of these small classes is preserved under action by the processor in question: the output is also in the same class as the input.
- Arbitrary signals outside of these preserved classes can be represented as a combination of members of the preserved classes.
- The combining process commutes with the processor: eg, additive combining commutes with linear systems.
- Thus, effectively, applying the processor directly upon the input signal is equivalent to a decomposition under the representation, applying the processor upon each component, and combining the results to get the output.
- The choice of a signal representation scheme is strongly dependent upon the kind of systems we are concerned with.

REPRESENTATION OF PERIODIC SIGNALS

- A continuous time signal $x(t)$ is periodic if there exists a positive and non zero T such that $x(t+T) = x(t): for\ all\ t$
- Given $x(t) = x_1(t) + x_2(t)$: $x_1(t) = x_1(t - T_1)$ & $x_2(t) = x_2(t - T_2)$
- Signal $x(t)$ will be periodic if
    - $\frac{T_1}{T_2} = n\ or\ \frac{1}{n}$ such that n is an integer and periodicity T = maximum (T_1, T_2)
    - $\frac{T_1}{T_2} = \frac{m}{n}$ then periodicity T = LCM (T_1, T_2)
- Signal $x(t)$ remains non periodic if $\frac{T_1}{T_2} \neq$ rational number
- Given a period T, another period T' is said to be harmonically related to it when T/T' is a positive integer.
- If a T-periodic signal is to be represented as a sum then,
    - All representing components must be periodic.
    - Additionally, all the periods of representing signals must be harmonically related to the period of $x(t)$.
- A preserved representation class for T-periodic signals in the context of processing by LTI system, will be of the form $\{ e^{j\frac{2\pi}{T}kt} : k = 1,2,3 \ldots \}$
    - The signal is a complex exponential and its form is preserved by any LTI system.
    - It is periodic and harmonically related to that of $x(t)$.

FOURIER SERIES

- It is a linear representation for periodic signals that are real or complex valued.
- Fourier series representation of a periodic signal $x(t)$ $(Synthesis)$:

$$x(t) = x(t-T) = \sum_{k=-\infty}^{\infty} x_k \, e^{-j\frac{2\pi}{T}kt}$$

- Fourier co-efficient x_k $(Analysis)$:

$$x_k = \frac{1}{T}\int_{0}^{T} x(t)\, e^{-j\frac{2\pi}{T}kt}\, dt$$

Convergence (in energy) of Fourier Series:

If $\int_0^T |x\,(t)|^2 dt \; < \infty \;:\; x(t) \text{ has finite power}$

⇒ all coefficients x_k will exist.

⇒ difference between $x(t)$ and its representation will have zero energy.

Convergence (pointwise) of Fourier series:

- Dirichlet conditions: If $x(t)$ is continuous at a point, the FS representation will converge
- Periodic signal $x(t)$ has a Fourier representation if it satisfies following Dirichlet conditions:
    - $x(t)$ is absolutely integrable over a period

$$\int_{0}^{T} |x\,(t)|dt \; < \infty$$

    - $x(t)$ has at most a finite number of maxima and minima over one period.
    - $x(t)$ has at most a finite number of discontinuities over a period and each discontinuity is finite.
- At a discontinuity, the FS representation converges to the midpoint of the discontinuity.

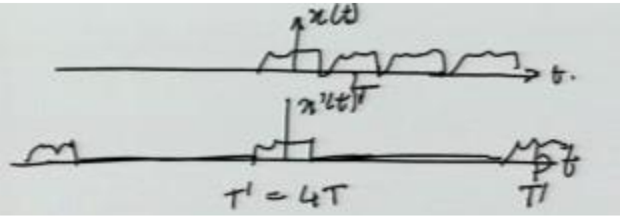
Properties of Fourier series:

- Fourier series is an orthogonal representation.
- Fourier series coefficients are generally complex numbers.
- If $x_k = x_{-k}^*$, then $x(t)$ is real.
- If $x_k = -x_{-k}^*$, then $x(t)$ is imaginary.

FOURIER SPECTRUM

- Fourier spectrum is plot of x_k, (complex amplitude) for each k^{th} harmonic.
- The Fourier spectrum is often called a line spectrum because at non-harmonic frequencies, the energy is zero.
- The lowest spectral line is located at $\omega = \omega_0$ which is the fundamental frequency of the spectrum.
- $\omega_0 = \frac{2\pi}{T}$, where T is period of a continuous time periodic signal.
- A spectrum can be finely resolved by changing the periodicity of the signal without changing its shape, padding it with extra zero space and thus increasing the period.

- For spectrum given below, Let $x(t)$ has $period\ T$ & $x'(t)$ has $period\ 4T$



- Spectrum of $x'(t)$ will be 4 times more finely resolved than that of $x(t)$ because its period T' is 4 times T, the period of $x(t)$
- As period $T \rightarrow \infty$, a continuous time periodic signal becomes non periodic and its spectrum becomes continuous.

FOURIER TRANSFORM

- Fourier transform is a Fourier representation of non-periodic signal $x(t)$.
- Analysis equation :

$$X(\omega) = \int_{t=-\infty}^{t=\infty} x\,(t)e^{-j\omega t}\,dt \qquad : \text{ forward transform}$$

- Synthesis equation:

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} X\,(\omega)e^{j\omega t}\,d\omega \;\; : \text{ inverse transform}$$

Continuous Time Fourier Transform (CTFT)

- It finds representations for non-periodic continuous time signals, in terms of complex exponentials that are themselves periodic but not harmonically related.
- The number of representing signals $e^{j\omega t}$ is uncountably infinite. Hence the linear combination that is the synthesis equation takes the form of an integral, rather than a sum.
- The representation is thus in the form of a continuous (not a line) spectrum and is generally complex valued (just like Fourier series) and has a real and imaginary components or magnitude and phase.

Convergence of Continuous Time Fourier Transform:

- Any continuous signal $x(t)$ which has finite energy can be represented in frequency domain via Fourier Transform which is known to converge in energy to the signal.

$$\int_{0}^{T} |x\,(t)|^2 dt \; < \infty$$

- The condition for pointwise convergence of $X(\omega)$ is the set of Dirichlet conditions:
  - $x(t)$ is absolutely integrable over a period

$$\int_{0}^{T} |x\,(t)| dt \; < \infty$$

  - $x(t)$ can have at most a finite number of maxima and minima over any finite interval.
  - $x(t)$ can have at most a finite number of discontinuities over a period and each discontinuity must be finite.
- Fourier Transform does not exist for finite power signals. It exists for finite energy signals only.
- Finite Power signal in time can be represented by Fourier transform if we allow impulses in the frequency domain.

PROPERTIES OF CONTINUOUS TIME FOURIER TRANSFORM

- The continuous-time Fourier transform may be viewed as a one-one relationship between complex valued functions of continuous time and complex valued functions of continuous frequency.

- Linearity:

$$x(t) \leftrightarrow X(\omega)$$
$$ax(t) + by(t) \leftrightarrow aX(\omega) + bY(\omega)$$

- Conjugation:

$$x^*(t) \leftrightarrow X^*(-\omega) \quad ; \; x^*(t) : \text{complex conjugate of x(t)}$$

- Time Reversal:

$$x(-t) \leftrightarrow X(-\omega)$$

- Frequency Reversal:

$$X(-\omega) \leftrightarrow x(-t)$$

- Time Scaling:

$$x(kt) \leftrightarrow \frac{1}{|k|} X\left(\frac{\omega}{k}\right)$$

- Frequency Scaling:

$$X\left(\frac{\omega}{k}\right) \leftrightarrow |k|x(kt)$$

- Symmetry:

  If $x(t)$ is real : $X(\omega)$ is conjugate symmetric

  If $x(t)$ is imaginary: $X(\omega)$is conjugate anti symmetric

  If $x(t)$ is real conjugate symmetric $x(t)$is real & even: $X(\omega)$is even & real

  If $x(t)$ is imaginary & conjugate antisymmetric: $X(\omega)$ is conjugate antisymmetric & imaginary

- Time Shift:

$$x(t-t_0) \leftrightarrow e^{-j\omega_0 t}X(\omega)$$

- Frequency Shift:

$$X(\omega-\omega_0) \leftrightarrow e^{j\omega_0 t}x(t)$$

- Convolution Property:

$$H(\omega).X(\omega) \leftrightarrow x(t) * y(t)$$

*PROPERTIES OF CONTINUOUS TIME FOURIER TRANSFORM

- Duality:

$$FT[X(t)] = 2\pi x(-\omega)$$

- Modulation property:

$$x(t).\,y(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * Y(\omega)$$

- Parseval Relation:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

- Differentiation:

$$\frac{dx(t)}{dt} = j\omega \, X(\omega)$$

- To determine running integral of any x(t)

$$\int_{-\infty}^{t} x(\tau) d\tau = x(t) * u(t) \;\; ; \;\; u(t) \text{ is unit step}$$

$$Y(\omega) = X(\omega).\,U(\omega) = \frac{X(\omega)}{j\omega} + X(\omega)\pi\delta(\omega) \;\; ; \;\; \pi\delta(\omega) = 0 \, ; \; \omega \neq 0$$

FREQUENCY RESPONSE OF CONTINUOUS LTI SYSTEM

- The output of an LTI system with the impulse response $h(t)$ due to input $x(t)$ is given by the convolution.

$$y(t) = x(t) * h(t)$$
$$\mathrm{Y}(\omega) = \mathrm{X}(\omega). H(\omega)$$
$$\text{Frequency Response } H(\omega) = \frac{\mathrm{Y}(\omega)}{\mathrm{X}(\omega)}$$

LTI System Characterized by Differential Equation:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) \quad : \; M \leq N$$

- Taking Fourier transform on both the sides

$$\sum_{k=0}^{N} a_k \, (j\omega)^k \mathrm{Y}(\omega) = \sum_{k=0}^{M} b_k \, (j\omega)^k \, \mathrm{X}(\omega)$$
$$\frac{\mathrm{Y}(\omega)}{\mathrm{X}(\omega)} = \frac{\sum_{k=0}^{M} b_k \, (j\omega)^k}{\sum_{k=0}^{N} a_k \, (j\omega)^k} = H(\omega)$$

- $H(\omega)$ is in polynomial rational form so it can be expanded into partial fractions.

Cascade form Implementation:

Factorizing $Y(\omega)\ \ and\ X(\omega)$: if the system is real,

$$H(\omega) = \frac{b_0}{a_0}\prod_{k=1}^{N/2} \frac{1+b_{1k}j\omega+b_{2k}(j\omega)^2}{1+a_{1k}j\omega+a_{2k}(j\omega)^2} \qquad : b_{1k}, b_{2k}, a_{1k}, a_{2k} \text{ are all real coefficients}$$

Parallel form Implementation:

Determination of Partial Fraction Expansion of $H(\omega)$

$$H(\omega) = H_0(\omega) + \sum_{k=1}^{N/2} H_k(\omega) \qquad : \qquad H_k(\omega) = \frac{b_{0k} + b_{1k}j\omega}{1 + a_{1k}j\omega + a_{2k}(j\omega)^2}$$

$$H_0(\omega) = \sum_{k=0}^{K} q_{0k}(j\omega)^k$$

DISCRETE SIGNALS AND SYSTEM REPRESENTATION

- Periodic Discrete Time sequences: $x[n] = x[n-N]$

Theory of representation

- Preserved classes of complex exponentials (discrete) that are harmonically related to $x[n]$ should be considered.
- A periodic complex exponential satisfies:

$$e^{j\frac{2\pi k}{N}n} = e^{j\frac{2\pi k}{N}(n-N)}$$

- The representation of the periodic signal $x[n] = x[n-N]$ can be done with no more than N distinct periodic complex discrete exponentials.
- Synthesis Equation:

$$x[n] = \sum_{k=0}^{N-1} x_k\, \varphi_k[n] \quad : \quad \varphi_k[n] = e^{j\frac{2\pi k}{N}n}$$

- Analysis Equation:

$$x_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k}{N}n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, \varphi_k^*[n]$$

Matrix form representation of Synthesis and Analysis equation

Synthesis Equation: $x[n] = \sum_{k=0}^{N-1} x_k\, \varphi_k[n]$

$$\begin{bmatrix} x_0 \\ \vdots \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} \varphi_0[0] & \varphi_1[0] & \dots\dots\dots & \varphi_{N-1}[0] \\ \varphi_0[0] & \varphi_1[1] & \dots\dots\dots & \varphi_{N-1}[1] \\ \vdots & \vdots & \dots\dots\dots & \vdots \\ \vdots & \vdots & \dots\dots\dots & \vdots \\ \varphi_0[N-1] & \varphi_1[N-1] & \dots\dots\dots & \varphi_{[N-1]}[N-1] \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ \vdots \\ \vdots \\ x[N-1] \end{bmatrix}$$

Analysis Equation: $x_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, \varphi_k^*[n]$

$$\begin{bmatrix} x_0 \\ \vdots \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} \varphi_0^*[0] & \varphi_0^*[1] & \dots\dots\dots & \varphi_0^*[N-1] \\ \varphi_1^*[0] & \varphi_1^*[1] & \dots\dots\dots & \varphi_0^*[N-1] \\ \vdots & \vdots & \dots\dots\dots & \vdots \\ \vdots & \vdots & \dots\dots\dots & \vdots \\ \varphi_{N-1}^*[0] & \varphi_{N-1}^*[1] & \dots\dots\dots & \varphi_{[N-1]}^*[N-1] \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ \vdots \\ \vdots \\ x[N-1] \end{bmatrix}$$

DISCRETE TIME FOURIER TRANSFORM:

The DTFT $X(\Omega)$ of a discrete time signal x[n] is a function of a continuous frequency Ω .

- Analysis Equation:

$$X(\Omega) = \sum_{n} x[n]\, e^{-j\Omega n}$$

- Synthesis Equation:

$$x[n] \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

- For $\sum_n |x[n]| \; < \infty$, $X(\Omega)$ will be finite for all $\Omega : 0 \leq \Omega \leq 2\pi$

PROPERTIES OF DISCRETE TIME FOURIER TRANSFORM:

- Discrete-time Fourier transform is one to one map of discrete time sequence and periodic continuous frequency functions.

- Linearity: $ax_1[n] + bx_2[n] \leftrightarrow aX_1(\Omega) + bX_2(\Omega)$

- Conjugation: $x^*[n] \leftrightarrow X^*(-\Omega)$

- Time Reversal: $x[-n] \leftrightarrow X(-\Omega)$

- Time/ Frequency Shift: $x[n - n_0] \leftrightarrow e^{-j\Omega n_0}X(\Omega)$

- Differencing & Summation:

  First difference of $x[n] = X(\Omega)\left(1 - e^{-j\Omega}\right)$

  Running sum of $X(\Omega)$: $\sum_{-\infty}^{n} x[n'] = X(\Omega).U(\Omega) = \frac{X(\Omega)}{(1-e^{-j\Omega})} + \pi X(\Omega)\sum_{l=-\infty}^{\infty} \delta(\Omega - 2\pi l)$

- Differentiation:

$$\frac{d}{d\Omega}X(\Omega) \leftrightarrow x[n](-jn)$$

- Symmetry Properties of the DTFT:

  $X(\omega)$is periodic with period 2π

  If $x(t)$ is real : $X(\omega)$ is conjugate symmetric

  If $x(t)$ is imaginary: $X(\omega)$is conjugate anti symmetric

  If $x(t)$ is real conjugate symmetric : $x(t)$is real & even: $X(\omega)$ is real & even

  If $x(t)$ is imaginary & conjugate anti symmetric: $X(\omega)$ is imaginary & conjugate anti symmetric

- Convolution:

  Linear convolution of two periodic signals is not possible. Thus periodic convolution of two periodic signal of same period, T:

$$x[n] * h[n] = \sum_{n'=0}^{N-1} x[n']h[n-n']$$

- Modulation Property:

$$x[n].y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega')Y(\Omega-\Omega')d\Omega'$$

FREQUENCY RESPONSE OF DISCRETE LTI SYSTEM

- The output of an LTI system with the impulse response $h[n]$ due to input $x[n]$ is given by the convolution.

$$\mathrm{y[n]} = \mathrm{x[n]} * \mathrm{h[n]}$$

$$\mathrm{Y}(\Omega) = \mathrm{X}(\Omega).H(\Omega)$$

Frequency Response $H(\Omega) = \frac{\mathrm{Y}(\Omega)}{\mathrm{X}(\Omega)}$

LTI System Characterized by Difference Equation:

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k] :\ M \leq N$$

- Taking Fourier transform on both the sides

$$\sum_{k=0}^{M} b_k\, e^{-jk\Omega} X(\Omega) = \sum_{k=0}^{N} a_k\, e^{-jk\Omega} Y(\Omega)$$

$$\frac{\mathrm{Y}(\Omega)}{\mathrm{X}(\Omega)} = \frac{\sum_{k=0}^{M} b_k\, e^{-jk\Omega}}{\sum_{k=0}^{N} a_k\, e^{-jk\Omega}} = H(\Omega)$$

- $H(\Omega)$ is in polynomial ration form so it can be expanded to partial fraction.

Cascade form Implementation:

Factorizing $\mathrm{Y}(\Omega)$ *and* $\mathrm{X}(\Omega)$

$$H(\Omega) = \frac{b_0}{a_0} \prod_{k=1}^{N/2} \frac{1 + b_{1k} e^{-j\Omega} + b_{2k} e^{-2j\Omega}}{1 + a_{1k} e^{-j\Omega} + a_{2k} e^{-2j\Omega}} \qquad : \ b_{1k}, b_{2k}, a_{1k}, a_{2k} \ \ are\ all\ real\ coefficient$$

Parallel form Implementation:

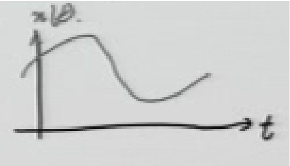
Determination of Partial Fraction Expansion of H(ω)

$$H(\Omega) = H_0(\Omega) + \sum_{k=1}^{N/2} H_k(\Omega) \qquad : \qquad H_k(\Omega) = \frac{b_{0k} + b_{1k} e^{-1j\Omega}}{1 + a_{1k} e^{-j\Omega} + a_{2k} e^{-2j\Omega}}$$

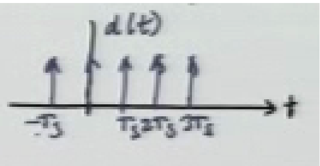
$$H_0(\omega) = \sum_{k=0}^{K} q_{0k} e^{-jk\Omega}$$

IDEAL SAMPLING

- When the sampling signal (impulse train) $d(t)$ is applied on a band limited signal $x(t)$, the sampled signal $x_s(t)$ obtained is given as:

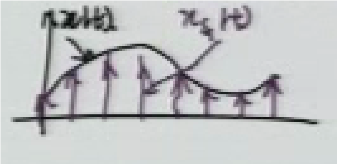


Input signal



Impulse train

$\xrightarrow{After\ Sampling}$



sampled signal

$$x_s(t) = x(t).d(t) = \sum_n x(nT_s).\delta(t - nT_s)$$

$$X_s(\omega) = \mathcal{F}[x(t).d(t)] = \frac{1}{2\pi}[X(\omega) * D(\omega)]$$

$$D(\omega) = \frac{2\pi}{T_s}\sum_k \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

$$X_s(\omega) = \frac{1}{T_s}\sum_k X(\omega - k\omega_s) \quad : \quad \omega_s = \frac{2\pi}{T_s}$$

- It is possible to retrieve original signal from its ideal sampled signal using an ideal low pass filter $[L(\omega)]$ of bandwidth B and gain T_s such that:

$$L(\omega) = \begin{cases} T_s \, ; |\omega| < \dfrac{B}{2} \\ 0 \, ; |\omega| \geq \dfrac{B}{2} \end{cases}$$

$$X_s(\omega).L(\omega) = \frac{1}{T_s}\sum_k X(\omega - k\omega_s)\,.L(\omega) = X(\omega)$$

- Sampling theorem states that signal $x(t)$ of bandwidth B can be reconstructed from its samples if sampling is done at frequency $\omega_s > 2B \; or \; T_s < \frac{\pi}{B}$
- If the signal is undersampled, i.e. $\omega_s < 2B$, the effect of spectral overlapping is called aliasing. When this occurs, the the reconstructed signal is distorted with respect to the original.

FLAT TOP SAMPLING

Problems with ideal sampling and retrieval:

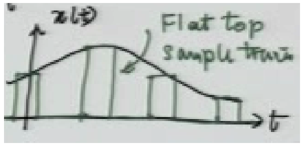
- Impulse trains are the ideal sampling signals, but they cannot be generated practically, so an ideal impulse train sampling is not feasible.
- Similarly, an ideal low pass filter is the perfect reconstruction agent, but is practically impossible to achieve: a real (finite order) lowpass filter has transition a band of non-zero width, which picks energy from neighbouring aliases..

Solution:

- Use of rectangular pulse train instead of an impulse train.
- There are two methods of sampling using rectangular pulse train:
    - Flat-top sampling
    - Faithful-top sampling
- Use of a hold circuit (finite order interpolator) instead of an ideal low pass filter.

Flat top sampling:

- Rectangular pulses assume a height equal to the value of $x(t)$ at the points of occurrence of the center of the rectangular pulses and their top will be flat.
- Mathematically, a convolution of ideal impulse sampled signal $x_s(t)$ with a rectangular pulse will result in flat top sampling.
- Rectangular pulse : $r_T(t) = \begin{cases} 1 \ ; \ |\mathrm{t} - \mathrm{nT_s}| \leq \mathrm{T} \\ 0 \quad ; \ elsewhere \end{cases}$
- Faithful top sampled train: $x_f(t) = \begin{cases} x(\mathrm{nT_s}) \ ; \ |\mathrm{t} - \mathrm{nT_s}| \leq \mathrm{T} \\ \quad 0 \quad ; \ elsewhere \end{cases}$



$$x_f(t) = x_s(t) * r_T(t)$$

$$X_f(\omega) = \frac{1}{2\pi} X_s(\omega) R_T(\omega)$$

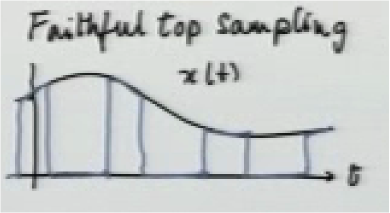
$$R_T(\omega) = \frac{2 \sin \omega T}{\omega}$$

To recover the original signal from the flat top sampled signal:

- An ideal low pass filter would not recover original signal without distortion.
- Due to flat top pulses the spectrum of the sampled signal is distorted.
- The narrower the pulse width (the more $r_T(t)$ resembles $\delta(t)$), and hence, the less distortion.

FAITHFUL TOP SAMPLING

- Pulses resulting from faithful-top sampling will follow exactly the values of the input signal during the ON time of each pulse in the train.
- Mathematically, a multiplication of the input signal with a rectangular pulse train is carried out, with the rectangular pulse train viewed as a convolution of a single rectangular pulse with an ideal unit impulse train.
- Faithful top sampled train: $X_t(t) = x(t).p(t)$
- Rectangular pulse train: $p(t) = \sum_n r_T(t - nT_s)$



$$X_t(t) = x(t).p(t) = \sum_n x(t) . r_T(t - nT_s)$$

$$X_t(\omega) = \mathcal{F}[x(t).p(t)] = \frac{1}{2\pi}[X(\omega) * P(\omega)]$$

$$P(\omega) = \frac{2\pi}{T_s}\sum_k \frac{2\sin\omega T}{\omega}\delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

$$X_t(\omega) = \frac{1}{T_s}\sum_k \frac{2\sin\omega T}{\omega}X(\omega - k\omega_s) \quad : \quad \omega_s = \frac{2\pi}{T_s}$$

To recover original signal from faithful top sampled signal:

- Filtering it with low pass filter $L(\omega) = \begin{cases} T_s\ ; |\omega| < \frac{\omega_s}{2} \\ 0\ ; |\omega| \geq \frac{\omega_s}{2} \end{cases}$

$$X_t(\omega).\,L(\omega) = 2X(\omega)$$

- Faithful top sampling followed by filtering it with ideal low pass filter will yield undistorted output $2X(\omega)$.

INTERPOLATION

- Ideal low pass filters cannot be constructed without investing infinite resources either in the form of hardware or computational effort.
- Problem can be overcome by use of hold circuit which is a non-ideal low pass filter..
- A desirable property of an interpolator system is that the output should coincide with the input at the data points given.
- First order hold circuit acts as a linear interpolator
- Both zeroth order hold $[h_0(t)]$ & first order hold $[h_1(t)]$ circuit uphold the above property of an interpolator because $h_i(t) = 0\ ;\ t = nT_s\ ;\ n \neq 0$
- But a second order hold circuit $h_2(t)$ onwards this property can be lost.
- First order circuit produces a series of ramps by connecting tops of these samples by straight lines.
- Ideal low pass filter acts as a hold circuit of infinite order, but maintains the property of the output of the interpolator coinciding with the data at given points by virtue of being self orthogonal for nonzero shifts that are multiples of T_s.

LAPLACE TRANSFORM:

- An absolutely integrable signal can be represented via its Fourier transform.
- Many important signals do not have Fourier transform, because the integral that defines $X(\omega)$ diverges.
- The remedy can be a Laplace transform of such signals in continuous time.
- For a continuous time signal $x(t)$ and complex frequency $s = (\sigma + j\omega)$ the Laplace transform $X(s)$ will be given as:

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt \quad : \; s = \sigma + j\omega$$

- It is a Fourier transform representation of continuous time signal $x(t)$ with a (possibly) decaying exponential term $e^{-\sigma t}$.

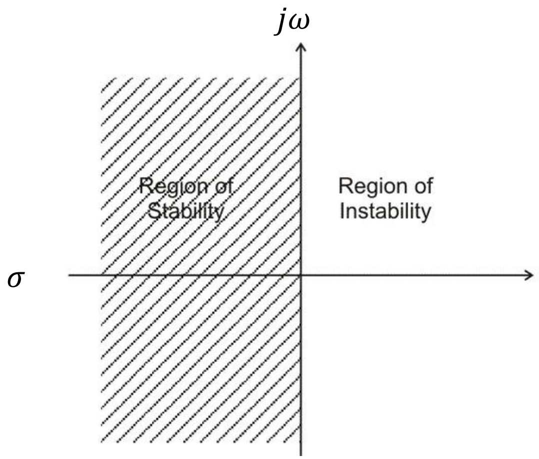
$$X(s) \overset{CTFT}{\longleftrightarrow} x(t)e^{-\sigma t}$$

- If Laplace transform of a causal $x(t)$ exists for σ_1, it will also exist for σ_2 where $\sigma_2 > \sigma_1$

- The range of values of complex variables s for which the Laplace Transform converges is called the Region of Convergence (ROC).

- Laplace Transform, a function of two variables, σ, ω is mapped on the so-called s plane.
- Real part of s is represented on the x-axis while Imaginary part of s on y-axis.
- The Fourier Transform of a signal is the Laplace Transform evaluated upon the $j\omega$ axis

Example of s-plane plot with ROC:



INVERSE LAPLACE TRANSFORM

- The inversion of Laplace transform is determination of $x(t)$ from its Laplace transform $X(s)$.

  - Inversion Formula:

$$x(t) = \frac{1}{2\pi j} \int_{s=\sigma-j\omega}^{s=\sigma+j\omega} X(s)e^{-st}\, ds :$$

  where σ is any point in the ROC of $X(s)$

  - By Partial Fraction expansion:
  If $X(s)$ is rational polynomial function of the form

$$X(s) = \frac{N(s)}{D(s)} = C\,\frac{(s-z_1)(s-z_2)\ldots\ldots\ldots..(s-z_n)}{(s-p_1)(s-p_2)\ldots\ldots\ldots\ldots(s-p_n)}$$

When $s \rightarrow z_n,\ X(s) \rightarrow 0;$ z_n are called zeros

When $s \rightarrow p_n,\ X(s) \rightarrow \infty;$ p_n are called poles

The form can be expanded into partial fraction. The simplified sum of $X(s)$ generated can be then be represented as inversion representation, where each term inverts to a standard time exponential.

Properties of ROC:

- ROC does not depend on ω but only on σ.
- Since the ROC is the set of points "s" at which $X(s)$ converges and since $X(s)$ clearly does not converge at $s = p_k$, it follows that no poles can lie within an ROC as that would violate the definition of ROC.
- If $x(t)$ has finite support and is absolutely integrable then its ROC is the entire s-plane.
- $x(t)$ is said to be left sided if there exists t_0 such that $x(t) = 0 \; for \; t > t_0$. E.g. - Anticausal signals.
- $x(t)$ is said to be right sided if there exists t_0 such that $x(t) = 0 \; for \; t < t_0$. E.g. - Causal signals.
- When $x(t)$ is right (left) sided and $x(t)e^{-\sigma t}$ is absolutely integrable at $s_0 = \sigma_0 + j\omega_0$ then the ROC of $x(t)$ extends indefinitely to the right (left) of s_0 and is called a "Rightward (Leftward) ROC".
- When $x(t)$ is two sided then it may be expressed as a sum of left and right sided signal components, and the rationale given above applied jointly upon the parts.

PROPERTIES OF LAPLACE TRANSFORM

- Linearity:

$$x_1(t) \rightarrow X_1(s) \ : R_1$$
$$x_2(t) \rightarrow X_2(s) \ : R_2$$
$$x(t) = x_1(t) + x_2(t) \rightarrow X(s) = X_1(s) + X_2(s) \ : R \supseteq R_1 \cap R_2$$

- Time Shift:

$$x(t - t_0) \leftrightarrow X(s)e^{-st_0} \quad : R$$

- Frequency Shift:

$$X(s - s_0) \leftrightarrow x(t)e^{s_0 t} \quad : \ R' = R + \sigma_0 \ ; \ s_0 = \sigma_0 + j\omega_0$$

- Time Scaling:

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right) \ : \ R' = \left\{\frac{\sigma}{a} + j\omega : \ \sigma + j\omega \subset R\right\}$$

- Convolution:

$$x_1(t) * x_2(t) \leftrightarrow X_1(s).X_2(s) \ : \ R \supseteq R_1 \cap R_2$$

- Differentiation in time:

$$x(t) = \frac{1}{2\pi j} \int_{s=\sigma - jw}^{s=\sigma + j\omega} X(s)e^{-st}\, ds$$

$$\frac{dx(t)}{dt} \leftrightarrow sX(s) \ : \ R' \supseteq R$$

- Running Integral:

$$\int_{-\infty}^{t} x(t')dt = u(t) * x(t) = \frac{1}{s}X(s) \ : \ R' = R \cap \{\sigma + j\omega; \sigma > 0\}$$

Z TRANSFORM

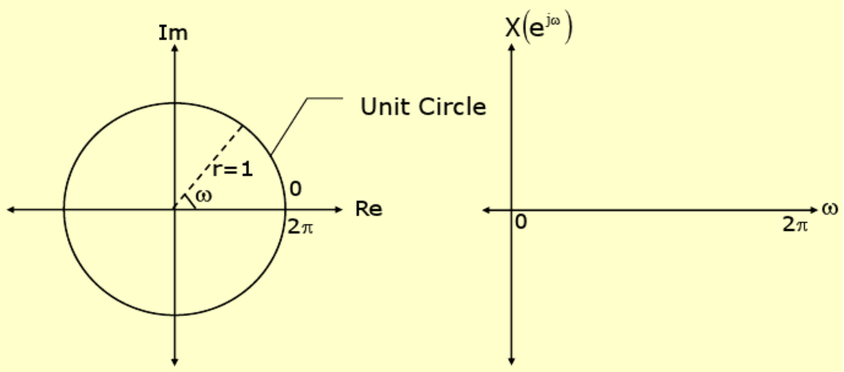
- DTFT does not exist if signal $x[n]$ is not absolutely summable.
- The remedy for the problem is Z transform.
- Z transform is the counterpart of Laplace Transform for discrete time signals.
- Z transform is to DTFT is what Laplace Transform is to CTFT.
- For a discrete time signal $x[n]$ and complex value $z = re^{j\omega}$; $r \ \epsilon \ [0, \infty]$ the Z transform $X(z)$ is given as:

$$X(z) = \sum_{n} x[n]z^{-n} \quad ; \ z = re^{j\Omega} \ for \ r \ \epsilon \ [0, \infty] \ \& \ \Omega \ \epsilon \ [0,2\pi]$$

$$X(z) \xleftrightarrow{DTFT} x[n]r^{-n}$$

- Z Transform, a function of two variables, r, Ω is mapped on the so-called z plane.
- Magnitude part of z is represented on the r-axis while phase part of z on θ-axis.
- The discrete Fourier Transform of a signal is the Z Transform evaluated upon $|r| = 1$, the so-called unit circle.

Example of z-plane plot with ROC:



- The set of values of z for which the z-transform converges is its Region of Convergence (ROC)
- Each value of r represents a circle of radius r.
- The region of convergence is made of circles.

INVERSE Z TRANSFORM

- The inversion of the Z transform is the determination of $x[n]$ from its Z transform $X(z)$.
  - Inversion Formula:

$$x[n] = \frac{1}{2\pi j} \int_{re^{-j\pi}}^{re^{j\pi}} X(z) z^{n-1}\, dz$$

  - Partial Fraction expansion:

    If $X(z)$ is of polynomial rational form

$$X(s) = \frac{N(z)}{D(z)} = C\,\frac{(z - z_1)(z - z_2) \dots \dots \dots \dots (z - z_n)}{(z - p_1)(z - p_2) \dots \dots \dots \dots (z - p_n)}$$

    Carry out a partial fraction expansion to obtain smaller terms that can be inverted from standard expressions. Assign to each component term an ROC such that the combined ROC of the component terms matches the ROC given for the overall expression.

Properties of ROC:

- The ROC of the Z Transform consists of complete circles about the origin.
- The ROC cannot contain any poles.
- If $x[n]$ has finite support $[n_a, n_b]$. then $x[n]$ is always summable and so will be $x[n]r^{-n}$. If the support of $x[n]$ includes points where n is negative, then $z = \infty$ will not lie in the ROC. If the support of $x[n]$ includes points where n is positive, then the origin will be a point of nonconvergence.
- If $x[n]$ is right (left) sided and the ROC of $X(z)$ contains the $z_0 = r_0 e^{j\Omega_0}$, then it will also include all points for z satisfying $|z|>|z_0|$ $(|z|>|z_0|)$. Further, if the support of $x[n]$ includes points where n is negative, then $z = \infty$ will not lie in the ROC. If the support of $x[n]$ includes points where n is positive, then the origin will be a point of nonconvergence.
- Two sided signals may be expressed as a sum of left sided and right sided signals. The ROC will be the intersection of the component ROCs if they both exist and intersect.

PROPERTIES OF Z TRANSFORM

- Linearity:

$$a x_1[n] + b x_2[n] \rightarrow a X_1(z) + b X_2(z) \quad : \quad R \supseteq R_1 \cap R_2$$

- Time Shift:

$$x[n-n_0] \leftrightarrow z^{-n_0} X(z) \quad : \quad R^{'} = R \text{ except possibly for } z = 0$$

- Frequency Shift:

$$X[z - z_0] \leftrightarrow {z_0}^{n} x[n] \quad : \quad R^{'} = R \text{ except for } z = 0$$

- Time Reversal:

$$x[n] \rightarrow X(z) \;:\; R$$

$$x[-n] \rightarrow X\left(\frac{1}{z}\right) \quad : \quad R^{'} = \frac{1}{R}$$

- Convolution:

$$x_1[n] * x_2[n] \rightarrow X_1(z).X_2(z) \quad : \quad R \supseteq R_1 \cap R_2$$

- Differentiation in the Z-domain:

$$\frac{dX(z)}{dz} = \sum_{n} x[n] z^{-n-1}(-n) \qquad : \quad R$$